# Permutations containing and avoiding certain patterns

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**Abstract**

*Let $T_k^m = \{\sigma \in S_k \mid \sigma_1 = m\}$. We prove that the number of permutations which avoid all patterns in $T_k^m$ equals $(k-2)!(k-1)^{n+1-k}$ for $k \leq n$. We then prove that for any $\tau \in T_k^1$ (or any $\tau \in T_k^k$), the number of permutations which avoid all patterns in $T_k^1$ (or in $T_k^k$) except for $\tau$ and contain $\tau$ exactly once equals $(n+1-k)(k-1)^{n-k}$ for $k \leq n$. Finally, for any $\tau \in T_k^m$, $2 \leq m \leq k-1$, this number equals $(k-1)^{n-k}$ for $k \leq n$. These results generalize recent results due to Robertson concerning permutations avoiding 123-pattern and containing 132-pattern exactly once.*

## 1 Introduction

In 1990, Herb Wilf asked the following: How many permutations, of length $n$, avoid a given pattern $\tau$? By pattern-avoiding we mean the following: A permutation $\alpha \in S_n$ avoids a permutation $\tau \in S_m$ if there is no $1 \leq i_1 < \ldots < i_m \leq n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_m})$ is order-isomorphic to $\tau = (\tau_1, \ldots, \tau_m)$. In this case, we write $\alpha \in S_n(\tau)$ and call the permutation $\tau$ a *pattern*. If a permutation $\alpha \in S_n$ avoids all patterns $\tau$ in a set $T$, we say that $\alpha$ avoids $T$ and write $\alpha \in S_n(T)$.

The first case to be examined was the case of permutations avoiding one pattern of length 3. Knuth [2] found that $|S_n(132)| = |S_n(123)| = c_n$ where $c_n$ is the $n$-th Catalan number given by the formula $c_n = \frac{1}{n+1}\binom{2n}{n}$, and it

is easy to prove that $|S_n(\tau)| = c_n$ for all $\tau \in S_3$. Later then Simion and Schmidt [6] found the cardinalities of $|S_n(T)|$ for all $T \subseteq S_3$.

A permutation $\alpha \in S_n$ contains $\tau \in S_m$ exactly $r$ times if there exist exactly $r$ different sequences $1 \le i_1^j < \ldots < i_m^j \le n$, $1 \le j \le r$, such that $(\alpha_{i_1^j}, \ldots, \alpha_{i_m^j})$ is order-isomorphic to $\tau = (\tau_1, \ldots, \tau_m)$ for all $j = 1, \ldots, r$.

Noonan shows in [3] that the number of permutations containing exactly one 123-pattern is given by a simple formula $\frac{3}{n}\binom{2n}{n+3}$. Bona [1] proves that expression $\binom{2n-3}{n-3}$ enumerates those permutations containing exactly one 132-pattern, and this result was extended by Robertson, Wilf and Zeilberger [5] to calculate the number of 132-avoiding permutations that have a given number of 123-patterns.

Robertson [4] proves that $(n-2)2^{n-3}$ gives the number of permutations containing exactly one 132-pattern and avoiding the 123-pattern; he also shows that the number of permutations containing exactly one 123-pattern and one 132-pattern equals $(n-3)(n-4)2^{n-5}$.

In this note we obtain a generalization of Robertson's result concerning the number 132-avoiding permutations that have one 123-pattern. As a byproduct we get a generalization of the following result due to Simion and Schmidt [6]: $|S_n(123, 132)| = |S_n(213, 231)| = 2^{n-1}$.

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## 2 Permutations avoiding $T_k^m$

Let $T_k^m = \{\sigma \in S_k \mid \sigma_1 = m\}$ for all $1 \le m \le k$, $k \in \mathcal{N}$. For example $T_3^1 = \{123, 132\}$ and $T_4^2 = \{2134, 2143, 2314, 2341, 2413, 2431\}$.

As a preparatory step, we calculate the cardinalities of the sets $S_n(T_k^m)$ for all $1 \le m \le k$, $k \in \mathcal{N}$.

**Theorem 2.1**

$$|S_n(T_k^m)| = (k-2)! \cdot (k-1)^{n+2-k},$$

for all $m, k, n \in \mathcal{N}$ such that $2 < k \le n$, $1 \le m \le k$.

**Proof** Let $G_n = S_n(T_k^m)$, and define the family of functions $f_h : S_n \to S_{n+1}$ by:

$$[f_h(\beta)]_i = \begin{cases} h, & \text{when } i = 1 \\ \beta_{i-1}, & \text{when } \beta_{i-1} < h \\ \beta_{i-1} + 1, & \text{when } \beta_{i-1} \geq h \end{cases}$$

for every $i = 1, ..., n+1$, $\beta \in S_n$ and $h = 1, ..., n+1$.

From this we see that:

if $\sigma \in G_n$ then $f_{n+1}(\sigma), f_n(\sigma), ..., f_{n+m-k+2}(\sigma), f_1(\sigma), ..., f_{m-1}(\sigma) \in G_{n+1}$

so $(k-1) \cdot |G_n| \leq |G_{n+1}|$ where $n \geq k$.

Assume that $(k-1) \cdot |G_n| < |G_{n+1}|$. Then there exists a permutation $\alpha \in G_{n+1}$ such that $m \leq \alpha_1 \leq n+m-k+1$, so there exist $k-1$ positions $1 < i_1 < \cdots < i_{k-1} \leq n+1$ such that the subsequence $\alpha_1, \alpha_{i_1}, \ldots, \alpha_{i_{k-1}}$ is order-isomorphic to one of the patterns in $T_k^m$, which contradicts the definition of $G_{n+1}$. So $(k-1) \cdot |G_n| = |G_{n+1}|$ for $n \geq k$. Besides $|G_k| = (k-1)(k-1)!$ (from the definition of this set), hence $|G_n| = (k-2)! \cdot (k-1)^{n+2-k}$ for $n \geq k$. ■

**Example 2.1** As a corollary we get the result of Simion and Schmidt [6]: $|S_n(T_3^1)| = |S_n(T_3^2)| = 2^{n-1}$ for all $n \in \mathcal{N}$. Other examples: $|S_n(T_4^1)| = 2 \cdot 3^{n-2}$ and $|S_n(T_5^2)| = 3 \cdot 2^{2n-5}$.

**Corollary 2.1**

$$|S_n(T_k^a \cup T_k^{a+1} \cup \ldots \cup T_k^b)| = (k-1)!(k+a-b-1)^{n+1-k},$$

where $1 \leq a \leq b \leq k$.

**Proof** Let $G_n = S_n(T_k^a \cup T_k^{a+1} \cup \ldots \cup T_k^b)$. From Theorem 2.1 we get that $\alpha \in G_n$ if and only if

$$f_1(\alpha), \ldots, f_{a-1}(\alpha), f_{n+b-(k-2)}(\alpha), \ldots, f_{n+1}(\alpha) \in G_{n+1}.$$

So $|G_{n+1}| = (k+a-b-1)|G_n|$. Besides $|G_k| = (k+a-b+1)(k-1)!$, hence the theorem holds. ■

For a general family $T_k^{i_1}, \ldots, T_k^{i_d}$ the situation is more complicated. However, the following recurrence can be proved.

**Corollary 2.2** For all $n \geq 2k+1$,

$$|S_n(T_k^{i_1} \cup \ldots \cup T_k^{i_d})| = (k + i_1 - i_d - 1)|S_{n-1}(T_k^{i_1} \cup \ldots \cup T_k^{i_d})|,$$

where $1 \leq i_1 < i_2 < \ldots < i_d \leq k$.

## 3 Permutations avoiding $T_k^1\backslash\{\tau\}$ and containing $\tau$ exactly once

Let $b_1 < \ldots < b_n$; we denote by $S_{\{b_1,\ldots,b_n\}}$ the set of all permutation of the numbers $b_1, \ldots, b_n$; for example, $S_{\{1,\ldots,n\}}$ is just $S_n$. As above we denote by $S_{\{b_1,\ldots,b_n\}}(T)$ the set of all permutations in $S_{\{b_1,\ldots,b_n\}}$ avoiding all the permutations in $T$.

**Proposition 3.1** Let $T \subseteq S_{\{c_1,\ldots,c_k\}}$. Then there exists $R \subseteq S_k$ such that $|S_n(R)| = |S_{\{c_1,\ldots,c_n\}}(T)|$.

**Proof** We define a function $f : S_{\{c_1,\ldots,c_k\}} \to S_k$ by

$$f((c_{i_1}, c_{i_2}, \ldots, c_{i_k})) = (i_1, i_2, \ldots, i_k),$$

then evidently $|S_{\{c_1,\ldots,c_n\}}(\tau)| = |S_n(f(\tau))|$ for all $\tau \in S_{\{c_1,\ldots,c_k\}}$.

Let $T = \{\tau_1, \ldots, \tau_l\}$ and $R = \{f(\tau_1), \ldots, f(\tau_l)\}$. So

$$S_{\{c_1,\ldots,c_n\}}(T) = \bigcap_{i=1}^{l} S_{\{c_1,\ldots,c_n\}}(\tau_i),$$

hence by the isomorphism $f$ we have that $|S_{\{c_1,\ldots,c_n\}}(T)| = |S_n(R)|$. ∎

**Defintion 3.1** Let $M_\tau^{k,m} = T_k^m\backslash\{\tau\}$, for $\tau \in T_k^m$. We denote by $S_n(T_k^m;\tau)$ the set of all permutations in $S_n$ that avoid $M_\tau^{k,m}$ and contain $\tau$ exactly once.

**Theorem 3.1**

$$|S_n(T_k^1;\tau)| = (n+1-k)\cdot(k-1)^{n-k},$$

for all $k \leq n$, $\tau \in T_k^1$.

**Proof** Let $\alpha \in S_n(T_k^1;\tau)$, and let us consider the possible values of $\alpha_1$:

1. $\alpha_1 \geq n-k+2$. Evidently $\alpha \in S_n(T_k^1;\tau)$ if and only if $\alpha \in S_{\{1,\ldots,n\}\backslash\{\alpha_1\}}(T_k^1;\tau)$.

2. $\alpha_1 \leq n-k$. Then there exist $1 < i_1 < \ldots < i_k \leq n$ such that $(\alpha_1, \alpha_{i_1}, \ldots, \alpha_{i_k})$ is a permutation of the numbers $n, \ldots, n-k+1, \alpha_1$. For any choice of $k-1$ positions out of $i_1, \ldots, i_k$, the corresponding permutations preceeded by $\alpha_1$ is order-isomorphic to some permutation in $T_k^1$. Since $\alpha$ avoids $M_\tau^{k,1}$, it is, in fact, order-isomorphic to $\tau$. We thus get at least $k$ occurrences of $\tau$ in $\alpha$, a contradiction.

3. $\alpha_1 = n-k+1$. Then there exist $1 < i_1 < \ldots < i_{k-1} \leq n$ such that $\eta = (\alpha_1, \alpha_{i_1}, \ldots, \alpha_{i_{k-1}})$ is a permutation of the numbers $n, \ldots, n-k+1$. As above, we immediately get that $\eta$ is order-isomorphic to $\tau$. We denote by $A_n$ the set of all permutations in $S_n(T_k^1; \tau)$ such that $\alpha_1 = n-k+1$, and define the family of functions $f_h : A_n \to S_{n+1}$ by:

$$[f_h(\beta)]_i = \begin{cases} 1, & \text{when } i = h \\ \beta_i + 1, & \text{when } i < h \\ \beta_{i-1} + 1, & \text{when } i > h \end{cases},$$

for every $i = 1, ..., n+1$, $\beta \in A_n$ and $h = 1, ..., n+1$. It is easy to see that for all $\beta \in A_n$,

$$f_{n+1}(\beta), \ldots, f_{n-k+3}(\beta) \in A_{n+1},$$

hence $(k-1)|A_n| \leq |A_{n+1}|$.

Now we define another function $g : A_{n+1} \to S_n$ by:

$$[g(\beta)]_i = \begin{cases} \beta_i - 1, & \text{when } i < h \\ \beta_{i+1} - 1, & \text{when } i + 1 > h \end{cases},$$

where $\beta_h = 1$, $i = 1, ..., n$, $\beta \in A_{n+1}$.

Observe that $h \geq n-k+3$, since otherwise already $(\beta_h, \beta_{h+1}, \ldots, \beta_{n+1})$ contains a pattern from $T_k^1$, a contradiction. It is easy to see that $g(\beta) \in A_n$ for all $\beta \in A_{n+1}$, hence $|A_{n+1}| \leq (k-1)|A_n|$. So finally, $|A_{n+1}| = (k-1)|A_n|$ and $|A_n| = (k-1)^{n-k}$, since $|A_k| = 1$.

Since the above cases $1, 2, 3$ are disjoint, and by Theorem 2.1 and Proposition 3.1 we obtain

$$|S_n(T_k^1; \tau)| = (k-1)|S_{n-1}(T_k^1; \tau)| + (k-1)^{n-k},$$

hence

$$|S_n(T_k^1; \tau)| = (n+1-k) \cdot (k-1)^{n-k},$$

for all $k \leq n$, $\tau \in T_k^1$. ■

**Example 3.1** $|S_n(123; 132)| = |S_n(132; 123)| = (n-2)2^{n-3}$, which is the result of Robertson in [4].

**Corollary 3.1**

$$|S_n(T_k^k;\tau)| = (n+1-k)\cdot(k-1)^{n-k},$$

for all $k \le n$, $\tau \in T_k^k$.

**Proof** Let $\beta$ be a permutation complement to $\tau$. By the natural bijection between the set $S_n(T_k^1;\beta)$ and the set $S_n(T_k^k,\tau)$ for all $\tau \in T_k^k$ we have that $S_n(T_k^k;\tau)$ have the same cardinality as $S_n(T_k^1;\beta)$, which is $(n+1-k)\cdot(k-1)^{n-k}$ by Theorem 3.1. ■

# 4 Permutations avoiding $T_k^m \backslash \{\tau\}$ and containing $\tau$ exactly once, $2 \le m \le k-1$

Now we calculate the cardinalities of the sets $S_n(T_k^m;\tau)$ where $2 \le m \le k-1$, $\tau \in T_k^m$.

**Theorem 4.1**

$$|S_n(T_k^m;\tau)| = (k-1)^{n-k},$$

for all $2 \le m < k \le n$, $\tau \in T_k^m$.

**Proof** Let $G_n = S_n(T_k^m;\tau)$, $\alpha \in G_n$, and let us consider the possible values of $\alpha_1$:

1. Let $\alpha_1 \le m-1$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2,\dots,\alpha_n) \in S_{\{1,\dots,n\}\backslash\{\alpha_1\}}(T_k^m;\tau)$.

2. $\alpha_1 \ge n-k+m+1$. Evidently $\alpha \in G_n$ if and only if $(\alpha_2,\dots,\alpha_n) \in S_{\{1,\dots,n\}\backslash\{\alpha_1\}}(T_k^m;\tau)$.

3. $m \le \alpha_1 \le n-k+m$. By definition we have that $|G_k| = 1$, so let $n \ge k+1$. If $m+1 \le \alpha_1$ then $\alpha$ contains at least $m \ge 2$ occurrences of a pattern from $T_k^m$, and If $\alpha_1 \le n-k+m-1$ then $\alpha$ conatins at least $k-m+1 \ge 2$ occurrences of a pattern from $T_k^m$, a contradiction.

Since the above cases $1,2,3$ are disjoint and by Proposition 3.1 we obtain $|G_n| = (k-1)|G_{n-1}|$ for all $k \le n$. Besides $|G_k| = 1$, hence $|S_n(T_k^m;\tau)| = (k-1)^{n-k}$. ■

**Example 4.1** $|S_n(213;231)| = |S_n(231;213)| = 2^{n-3}$.

## References


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